MSG800/MVE170 Basic Stochastic Processes

Written exam Thursday 12 April 2012 8.30 am - 12.30 am

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Aids: Either two A4-sheets (4 pages) of hand-written notes (xerox-copies and/or computer print-outs are not allowed) or Beta (but not both these aids).

Grades: 12 points (40%) for grades 3 and G, 18 points (60%) for grade 4, 21 points (70%) for grade VG and 24 points (80%) for grade 5, respectively.

Motivations: All answers/solutions must be motivated.

Good Luck!

Task 1. Let $\{X(t)\}_{t\in\mathbb{R}}$ be a continuous time normal (=Gaussian) random process. Which of the following five random processes are also normal: $\{X(2t)\}_{t\in\mathbb{R}}$, $\{2X(t)\}_{t\in\mathbb{R}}$, $\{X(t)^2\}_{t\in\mathbb{R}}$, $\{X(t)-2\}_{t\in\mathbb{R}}$ and $\{X(t)-X(0)\}_{t\in\mathbb{R}}$? (The answer must be fully motivated!) **(5 points)**

Task 2. Let $\{X(t)\}_{t\geq 0}$ be a continuous time martingale. Which of the following five random processes are also martingales: $\{X(2t)\}_{t\geq 0}$, $\{2X(t)\}_{t\geq 0}$, $\{X(t)^2\}_{t\geq 0}$, $\{X(t)-2\}_{t\geq 0}$ and $\{X(t)-X(0)\}_{t\geq 0}$? (The answer must be fully motivated!) **(5 points)**

Task 3. Let $\{X(t)\}_{t\in\mathbb{R}}$ be a continuous time strict-sense stationary random process. Which of the following five random processes are also strict-sense stationary random processes: $\{X(2t)\}_{t\in\mathbb{R}}$, $\{2X(t)\}_{t\in\mathbb{R}}$, $\{X(t)^2\}_{t\in\mathbb{R}}$, $\{X(t)-2\}_{t\in\mathbb{R}}$ and $\{X(t)-X(0)\}_{t\in\mathbb{R}}$? Also, answer the same question when strict-sense stationarity is replaced with wide-sense stationarity (at both occurances). (The answer must be fully motivated!) **(5 points)**

Task 4. Let $\{X(t)\}_{t\in\mathbb{R}}$ be a continuous time wide-sense stationary random process. Which of the following five random processes can be created as the output process from a continuous time linear time-invariant system with $\{X(t)\}_{t\in\mathbb{R}}$ as its input process: $\{X(2t)\}_{t\in\mathbb{R}}$, $\{2X(t)\}_{t\in\mathbb{R}}$, $\{X(t)^2\}_{t\in\mathbb{R}}$, $\{X(t)-2\}_{t\in\mathbb{R}}$ and $\{X(t)-X(0)\}_{t\in\mathbb{R}}$? (The answer must be fully motivated!) **(5 points)**

Task 5. Consider a homogeneous Markov chain $\{X_n,\, n\geq 0\}$ with state space (/possible values) E, initial state probability vector $\mathbf{p}(0)$ and transition matrix P given by

$$E=\{0,1,2\},\qquad \mathbf{p}(0)=[1\ \ 0\ \ 0]\qquad \text{and}\qquad P=\begin{bmatrix}1/3 & 1/3 & 1/3\\ 1/3 & 1/3 & 1/3\\ 1/3 & 1/3 & 1/3\end{bmatrix},$$

respectively. Find the expected value $\mathbf{E}\{T_0\}$ of the time $T_0 = \min\{n \geq 1 : X_n = 0\}$ it takes for the chain to come back to its starting state 0. **(5 points)**

Task 6. Consider a M/M/1/2 queueing system with mean arrival rate $\lambda = 1$ and mean service rate $\mu = 1$. Write a computer programme that by means of stochastic simulation finds an approximative value of the average length of a busy period of the queueing system (that is, a period during which the server is busy). **(5 points)**

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Solutions to written exam Thursday 12 April

Task 1. The processes number 1-2 and 4-5 are normal because for each of them it holds that any vector of process values is jointly normal distributed. Process number 3 is not normal because not even a single process value $X(t)^2$ is normal distributed.

Task 2. The processes number 2 and 4-5 are martingales as for each of them it is easy to see that the martingale property Eq. 5.88 in Hsu's book holds. For process number 1 we have $\mathbf{E}\{X(2t)\,|\,\mathcal{F}_s\} = X(s)$ for $0 \leq s \leq t$, so that the process is not a martingale unless $X(s) = X(2s)$ for $s \geq 0$. For process number 3 we have $\mathbf{E}\{X(t)^2|\mathcal{F}_s\} = \mathbf{E}\{(X(t) - X(s))^2|\mathcal{F}_s\} + 2\,\mathbf{E}\{(X(t) - X(s))\,X(s)\,|\,\mathcal{F}_s\} + \mathbf{E}\{X(s)^2|\mathcal{F}_s\} = \mathbf{E}\{(X(t) - X(s))^2|\mathcal{F}_s\} + 2\,\mathbf{E}\{X(t) - X(s)\,|\,\mathcal{F}_s\}\,X(s) + X(s)^2 = \mathbf{E}\{(X(t) - X(s))^2|\mathcal{F}_s\} + X(s)^2$ for $0 \leq s \leq t$, so that the process is not a martingale unless $\mathbf{E}\{(X(t) - X(s))^2|\mathcal{F}_s\} = 0$ for $0 \leq s \leq t$. However, in the latter case we have $\mathbf{E}\{(X(t) - X(s))^2\} = \mathbf{E}\{\mathbf{E}\{(X(t) - X(s))^2|\mathcal{F}_s\}\} = 0$ for $0 \leq s \leq t$, which is to say that the process must satisfy $X(t) = X(0)$ for all $t \geq 0$. In other words, process number 3 is a martingale if and only if $X(t) = X(0)$ for all $t \geq 0$.

Task 3. As for the first question, all the five processes are strict-sense stationary as for each of them it is easy to see that the defining property for strict-sense stationarity Eq. 5.14 in Hsu's book holds. As for the second question, the processes number 1-2 and 4-5 all are wide-sense stationary as for each of them it is easy to see that the defining properties for wide-sense stationarity Equations 5.21-5.22 in Hsu's book hold, while it is impossible to judge whether process number 3 is wide-sense stationary as we have no available information about forth order moments like $\mathbf{E}\{X(s)^2 X(t)^2\}$.

Task 4. The possible outputs of an LTI system are made up of all possible linear combinations of the input. The only two of the five mentioned processes that are such are the processes number 2 and 5.

Task 5. We have $\mathbf{E}\{T_0\} = \mathbf{E}\{\hat{T}_0\}$ where $\{\hat{X}_n,\, n \geq 0\}$ is a Markov chain with state space $\hat{E}$, initial state probability vector $\hat{\mathbf{p}}(0)$ and transition matrix $\hat{P}$ given by

$$\hat{E} = \{0, 1\}, \qquad \hat{\mathbf{p}}(0) = [1 \;\; 0] \qquad \text{and} \qquad P = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{bmatrix},$$

respectively, and where $\hat{T}_0 = \min\{n \geq 1 : \hat{X}_n = 0\}$. Here basic probability theory gives $\mathbf{E}\{\hat{T}_0\} = (1/3) \cdot 1 + (2/3) \cdot \big(1 + \sum_{n=1}^{\infty} n\,(2/3)^{n-1}(1/3)\big) = \ldots = 3$.

Task 6.

```
Clear[aver,reps,que]; reps=100000;
For[i=1; aver=0, i<=reps, i++, que=1; While[que>=1,
  If[que==3, aver=aver+Random[ExponentialDistribution[1]]; que=2];
  If[que>=1&&que<=2, aver=aver+Random[ExponentialDistribution[2]];
    If[Random[UniformDistribution[0,1]]<=1/2, que=que-1, que=que+1]]]];
aver/reps
```